Journal of Algebra **214,** 400–417 (1999)
Article ID jabr.1998.7717, available online at http://www.idealibrary.com on IDEAL®

# Multiplier Hopf Algebras of Discrete Type

Alfons Van Daele



and

Yinhuo Zhang

*Department of Mathematics, University of Antwerp, Universiteitsplein 1, B-2610 Wilrijk, Belgium*
E-mail: Zhang@uia.ua.ac.be

*Communicated by Susan Montgomery*

Received January 17, 1998

In this paper, we study regular multiplier Hopf algebras with cointegrals. They are a certain class of multiplier Hopf algebras, still sharing many nice properties with the (much smaller class of) finite-dimensional Hopf algebras. Recall that a multiplier Hopf algebra is a pair $(A, \Delta)$ where $A$ is an algebra over $\mathbb{C}$, possibly without identity, and $\Delta$ is a comultiplication on $A$ (a homomorphism of $A$ into the multiplier algebra $M(A \otimes A)$ of $A \otimes A$) satisfying certain properties. The typical example is the algebra $A$ of complex functions with finite support in a group $G$, with pointwise multiplication and where the comultiplication is defined by $(\Delta f)(p, q) = f(pq)$ whenever $f \in A$ and $p, q \in G$. A left cointegral in a multiplier Hopf algebra is an element $h \in A$ such that $ah = \epsilon(a)h$ for all $a \in A$ where $\epsilon$ is the counit of $A$. In the group example, this is the function that is 1 on the identity of the group and 0 everywhere else. In this paper, we show that cointegrals are unique (up to a constant) if they exist and that they are faithful. We also show that on a regular multiplier Hopf algebra with a left cointegral, there exists also a left integral. Recall that a left integral is a linear functional $\varphi$ on $A$ such that $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ where $\iota$ is the identity map (and where the equation is to be considered in $M(A)$). A multiplier Hopf algebra with cointegrals is therefore an algebraic quantum group of discrete type. We will also obtain different necessary and sufficient conditions on the algebra $A$ for a multiplier Hopf algebra $(A, \Delta)$ to have cointegrals (i.e., to be of discrete type). The algebras turn out to be Frobenius, quasi-Frobenius, and Kasch. 



0021-8693/99 $30.00

## 1. INTRODUCTION

Let us first recall the notion of a multiplier Hopf algebra (see [10]).

Let $A$ be an algebra over $\mathbb{C}$ with a non-degenerate product. Remark that $A$ may or may not have an identity. The multiplier algebra $M(A)$ is defined as the set of pairs $(\rho_1, \rho_2)$ of linear maps of $A$ into $A$ satisfying $\rho_2(a)b = a\rho_1(b)$ for all $a, b \in A$. If we set $x = (\rho_1, \rho_2)$, then we say that $\rho_1$ is left multiplication with $x$ and $\rho_2$ is right multiplication with $x$. We write $\rho_1(a) = xa$ and $\rho_2(a) = ax$. The condition relating $\rho_1$ and $\rho_2$ is nothing else but associativity $(ax)b = a(xb)$. It is easy to see that this set $M(A)$ can be made into an algebra, that it has an identity, and that $A$ has a natural imbedding as a dense (essential) two-sided ideal in $M(A)$. In fact, $M(A)$ can be characterized as the largest algebra with identity in which $A$ sits as a dense two-sided ideal.

Now consider the tensor product algebra $A \otimes A$. The product in $A \otimes A$ is again non-degenerate and we can consider the multiplier algebra $M(A \otimes A)$. There are natural imbeddings

$$A \otimes A \subseteq M(A) \otimes M(A) \subseteq M(A \otimes A).$$

A comultiplication on $A$ is a homomorphism $\Delta\colon A \to M(A \otimes A)$ such that:

(i) $\Delta(A)(1 \otimes A)$ and $(A \otimes 1)\Delta(A)$ are subsets of $A \otimes A$,

(ii) $(a \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$ for all $a, b, c \in A$.

In (ii), we use $\iota$ for the identity map. This condition is called the coassociativity of $\Delta$. Remark that we need (i) to give a meaning to (ii).

A pair $(A, \Delta)$ of an algebra $A$ with a non-degenerate product and a coproduct $\Delta$ on $A$ is called a multiplier Hopf algebra if the maps

$$a \otimes b \to \Delta(a)(1 \otimes b) \qquad \text{and} \qquad a \otimes b \to (a \otimes 1)\Delta(b)$$

extend to bijective linear maps from $A \otimes A$ to itself. A multiplier Hopf algebra is called regular if also $(A, \Delta')$ is a multiplier Hopf algebra where $\Delta'$ is the opposite comultiplication obtained from $\Delta$ by composing $\Delta$ with the flip.

Any Hopf algebra is a multiplier Hopf algebra. Conversely, if $(A, \Delta)$ is a multiplier Hopf algebra and if $A$ has an identity, then $A$ is a Hopf algebra. So, the notion of a multiplier Hopf algebra is a natural extension of the notion of a Hopf algebra for algebras without identity.

And indeed, just as for Hopf algebras, we have the existence of a counit and an antipode for multiplier Hopf algebras. The counit is the unique linear map $\epsilon\colon A \to \mathbb{C}$ such that

$$(\epsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab \qquad \text{and} \qquad (\iota \otimes \epsilon)((a \otimes 1)\Delta(b)) = ab$$

for all $a, b \in A$. It is a homomorphism. The antipode is the unique linear map $S: A \to M(A)$ satisfying

$$m(S \otimes \iota)(\Delta(a)(1 \otimes b)) = \epsilon(a)b \qquad \text{and}$$
$$m(\iota \otimes S)((a \otimes 1)\Delta(b)) = a\epsilon(b)$$

for all $a, b \in A$ (where $m$ denotes the multiplication, considered as a linear map from $A \otimes A$ to $A$ and extended to $M(A) \otimes A$ and $A \otimes M(A)$). The antipode is a anti-homomorphism. If the multiplier Hopf algebra is regular, then $S$ maps $A$ into itself and it is bijective.

The example, motivating the introduction of this generalization of a Hopf algebra, comes from a group in the following way. Let $G$ be any group and let $A$ be the vector space of complex functions with finite support in $G$. Then $A$ is made into an algebra by taking pointwise product. This product is non-degenerate. The multiplier algebra $M(A)$ is canonically identified with the algebra of all complex functions on $G$. The tensor product $A \otimes A$ is identified with the algebra of complex functions with finite support in $G \times G$. The multiplier algebra $M(A \otimes A)$ is the algebra of all complex functions on $G \times G$. The comultiplication $\Delta$ is defined by $(\Delta f)(p, q) = f(pq)$ whenever $f \in A$ and $p, q \in G$. It is not hard to verify that $(A, \Delta)$ is indeed a (regular) multiplier Hopf algebra, that the counit is given by $\epsilon(f) = f(e)$ where $e$ is the unit of $G$, and that the antipode $S$ is given by $(Sf)(p) = f(p^{-1})$. Also remark that $(A, \Delta)$ is not a Hopf algebra when $G$ is infinite (as $A$ has no identity in that case).

In what follows, $(A, \Delta)$ will be any regular multiplier Hopf algebra.

By $A'$ we will denote the space of linear functionals on $A$. When $\omega \in A'$, we can define elements $(\omega \otimes \iota)\Delta(a)$ and $(\iota \otimes \omega)\Delta(a)$ in $M(A)$ by

$$((\omega \otimes \iota)\Delta(a))b = (\omega \otimes \iota)(\Delta(a)(1 \otimes b)$$
$$a((\iota \otimes \omega)\Delta(b)) = (\iota \otimes \omega)((a \otimes 1)\Delta(b))$$

for any $a, b \in A$.

A non-zero linear map $\varphi$ on $A$ is called a *left integral* if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all $a \in A$. Similarly, a non-zero linear map $\psi$ on $A$ is called a *right integral* if $(\psi \otimes \iota)\Delta(a) = \psi(a)1$ for all $a \in A$. Such integrals do not always exist. But it is easy to see that the antipode will convert a left integral to a right integral and a right to a left one. In the group example above, the map $f \to \sum_p f(p)$ is both a left and a right integral. In general, however, left and right integrals need not be the same.

In [12, 13], regular multiplier Hopf algebras with integrals have been studied. It is shown that left (resp. right) integrals are unique (up to a scalar) if they exist. They are also *faithful* (i.e., if $a \in A$ and if $\varphi(xa) = 0$ for all $x \in A$, then $a = 0$ and if $\varphi(ax) = 0$ for all $x$, also $a = 0$). It is also shown that left (resp. right) integrals are K.M.S. functionals in the sense that, e.g., the spaces of linear functionals $\{\varphi(a\,\cdot\,) \mid a \in A\}$ and $\varphi(\cdot a) \mid a \in A\}$ coincide. Moreover, this subspace of $A'$ can be made into a regular multiplier Hopf algebra by taking the adjoints of the comultiplication and the multiplication in $A$. This multiplier Hopf algebra is called the dual of $(A, \Delta)$ and it is denoted by $(\hat{A}, \hat{\Delta})$. It is also possible to prove a duality theorem in this context, extending the famous Pontryagin duality between abelian discrete and abelian compact groups. In this paper, we will call a regular multiplier Hopf algebra with integrals an *algebraic quantum group*.

Let us now introduce the notion of cointegrals. A *left cointegral* in a regular multiplier Hopf algebra is a non-zero element $h \in A$ such that $ah = \epsilon(a)h$ for all $a \in A$. Similarly, a *right cointegral* is a non-zero element $k \in A$ such that $ka = \epsilon(a)k$ for all $a$. Again, such cointegrals do not always exist. But also here, the antipode will turn a left cointegral into a right one and a right one into a left one. Also in the group example above, the function that is 1 in the unit $e$ of the group and 0 everywhere else will be a left and a right cointegral. In general, however, left and right cointegrals are different (see, e.g., Example 3.10).

In this paper (Section 2), we prove that left (resp. right) cointegrals are unique (up to a scalar) if they exist. They are also *faithful* in the sense that if $\omega \in A'$ and $(\iota \otimes \omega)\Delta(h) = 0$, then $\omega = 0$ and if $(\omega \otimes \iota)\Delta(h) = 0$, then also $\omega = 0$. This faithfulness property is basic for the main result in Section 2. This says that if cointegrals exist, then also integrals exist. It follows that a multiplier Hopf algebra with cointegrals is an algebraic quantum group of discrete type in the sense of Definition 5.2 of [13].

The existence proof of the integral that we give here for regular multiplier Hopf algebras with cointegrals is similar to the existence proof in the case of discrete quantum groups [11]. It is, however, more general (and slightly simpler). Recall that a *discrete quantum group* is defined in [11] as a regular multiplier Hopf algebra $(A, \Delta)$ where the underlying algebra $A$ is a direct sum of full matrix algebras. In fact, only the $^*$-algebra case was considered in [11].

This paper is, also from other points of view, a generalization of the theory of discrete quantum groups, as first introduced in [4] and independently developed in [11]. In [11], it is shown that cointegrals and integrals exist when $(A, \Delta)$ is a regular multiplier Hopf $^*$-algebra where $A$ is a direct sum of full matrix algebras over $\mathbb{C}$. In Section 3 of this paper, we will give several other properties of the underlying algebra that are necessary and sufficient for $(A, \Delta)$ to be of discrete type (i.e., to have cointegrals).

The conditions are Frobenius, quasi-Frobenius, and Kasch. All these are properties of the underlying algebra and have nothing to do with the comultiplication.

One of the main tools in all this is related with some module properties. In [14], we develop a theory of actions of multiplier Hopf algebras. In the beginning of that paper, we first have to set some terminology about modules. In this paper, we consider the space $A'$ of all linear functionals on $A$ as a left $A$-module where the action of $A$ on $A'$ is defined as usual by $a\omega = \omega(\cdot\, a)$ whenever $a \in A$ and $\omega \in A'$. We denote by $H$ the submodule $AA'$. Remark that this coincides with $A'$ if $A$ has an identity, but that it will be a proper submodule in general. It now turns out that, given a left cointegral $h$, the map $\omega \to (\iota \otimes (\omega \circ S))\Delta(h)$ is a $A$-module isomorphism of the left $A$-module $H$ to the left $A$-module $A$ (with left multiplication as module action). This property means essentially that $A$ is a Frobenius algebra. But it is also used to give the other characterizations in terms of ideals in $A$.

We have already discussed most of the terminology in this introduction. Let us now give some standard references and a comment on the use of the Sweedler notation for multiplier Hopf algebras in this paper.

For Hopf algebras, we refer to [1, 8]. For multiplier Hopf algebras, [10] is the basic reference. Algebraic quantum groups (i.e., regular multiplier Hopf algebras with integrals) are treated in [12, 13] and the discrete quantum groups in [4, 11].

In [3], the use of the Sweedler notation for regular multiplier Hopf algebras has been introduced. We will also use this notation in this paper. We will, e.g., write

$$\sum_{(a)} a_{(1)} \otimes a_{(2)} b \qquad \text{for } \Delta(a)(1 \otimes b)$$

and

$$\sum_{(a)} b a_{(1)} \otimes a_{(2)} \otimes a_{(3)} c \qquad \text{for } (b \otimes 1 \otimes 1)(\iota \otimes \Delta)(\Delta(a)(1 \otimes c)),$$

etc. One has to make sure that at least all but one factor $a_{(k)}$ is *covered* by an element in $A$. The reader should be aware of the fact that there is some danger in using the Sweedler notation in this setting. On the other hand, many formulas become much more transparent. And, with some care, it is always possible to write out the proofs using Sweedler notation in detail without using it. Moreover, the result that we proved in [14] and that we also discuss in Section 2, saying that given elements $a_1, a_2, \ldots, a_n$ in any multiplier Hopf algebra $A$, there exist elements $e, f \in A$ such that $ea_k = a_k f = a_k$ for all $k$, is very useful and provides another strong justification of the use of Sweedler's notation (see the discussion in [14] and in Section 2 of this paper).

## 2. MULTIPLIER HOPF ALGEBRAS WITH COINTEGRALS

Let $(A, \Delta)$ be a regular multiplier Hopf algebra. Denote by $\epsilon$ the counit and by $S$ the antipode. Remark that $S$ maps $A$ to $A$ and that it is bijective (because we assume $A$ to be regular).

Now, let us first recall the following property, true for any regular multiplier Hopf algebra.

2.1. PROPOSITION. *Given $a_1, a_2, \ldots, a_n \in A$, there exists elements $e, f \in A$ such that $ea_i = a_i$ and $a_i f = a_i$ for all $i$.*

The proof is not very hard and can be found in [14]. It goes as follows. Assume that no such element $e$ exists. Then there exist linear functionals $\omega_1, \omega_2, \ldots, \omega_n \in A'$ so that $\sum \omega_i(ea_i) = 0$ for all $e \in A$ but $\sum \omega_i(a_i) \neq 0$. However, if $\sum \omega_i(ea_i) = 0$ for all $e \in A$, then $\sum_{i,(b)} \omega_i(cb_{(2)}a_i)b_{(1)} = 0$ for all $b, c$. Now we replace $c$ by $S(b_{(1)})$ to get $\sum \omega_i(a_i) = 0$. This last step is justified as we will see, in a similar situation, in the proof of Proposition 2.5 below where we will be more precise. The existence of the element $f$ is proven in a similar way or by using the antipode.

When $A$ is an algebraic quantum group, i.e., when $A$ has integrals, then it is possible to take $e = f$ (see [14]). In Section 3, we will prove an even stronger result when $A$ is of discrete type. In that case it will be possible to assume moreover that $e$ is idempotent (see Proposition 3.1). It is not known if a result, stronger than the one in Proposition 2.1, is true for any (regular) multiplier Hopf algebra. But for many applications, the property in Proposition 2.1 is enough anyway.

The result in Proposition 2.1 can, e.g., be used to justify the use of the Sweedler notation in the following sense. Suppose that we want to write $\sum_{(a)} a_{(1)} \otimes a_{(2)} b$ for $\Delta(a)(1 \otimes b)$. Because we have an element $e \in A$ such that $eb = b$, we have $\Delta(a)(1 \otimes b) = (\Delta(a)(1 \otimes e))(1 \otimes b)$ and we can think of $\sum_{(a)} a_{(1)} \otimes a_{(2)}$ as $\Delta(a)(1 \otimes e)$ (in $A \otimes A$) (see [14] for a more detailed discussion).

Observe that it follows from Proposition 2.1 that $A^2 = A$, a result that we also will use occasionally. But this property is also an easy consequence of the axioms of a multiplier Hopf algebra.

In Section 3, we will consider the dual space $A'$ as a left $A$-module for the natural action of $A$ defined by $a\omega = \omega(\cdot\, a)$ whenever $a \in A$ and $\omega \in A'$. We will also work with the $A$-submodule $AA'$. Using Proposition 2.1, we can prove the following result about this submodule.

2.2. Lemma. *We have* $AA' = \{\omega(\cdot\, a) \mid a \in A,\, \omega \in A'\}$.

*Proof.* We have to show that the right hand side is a subspace of $A'$. To see this, take $a, b \in A$ and $\omega, \psi \in A'$ and define $\rho \in A'$ by $\rho(c) = \omega(ca) + \psi(cb)$ for all $c \in A$. Use Proposition 2.1 to get an element $e \in A$ such that $ea = a$ and $eb = b$. Then $\rho = \rho(\cdot\, e)$. ∎

We will denote this submodule $AA'$ of $A'$ by $H$. Remark that $H$ is also an algebra for the product dual to the coproduct (see, e.g., [10]). The counit belongs to $H$ (take any $a$ so that $\epsilon(a) = 1$, then $\epsilon = \epsilon(a\,\cdot))$ and it is the identity in $H$. In this section we will only use $H$ as subspace of $A'$.

It also follows from Proposition 2.1 that elements in $H$ have unique extensions to $M(A)$ satisfying $\rho(x) = \omega(xa)$ when $\rho = \omega(\cdot\, a)$. Indeed, suppose that $a, b \in A$ and $\omega, \psi \in A'$ and $\omega(ca) = \psi(cb)$ for all $c \in A$. Take $e \in A$ such that $ea = a$ and $eb = b$. For all $x \in M(A)$ we will have $\omega(xa) = \omega((xe)a) = \psi((xe)b) = \psi(xb)$. This property will be used frequently in this paper.

We are now ready to give the main definition of this paper (see also Definition 5.2 of [13]).

2.3. Definition. A *left cointegral* in $A$ is a non-zero element $h \in A$ such that $ah = \epsilon(a)h$ for all $a \in A$. Similarly, a *right cointegral* in $A$ is a non-zero element $k \in A$ so that $ka = \epsilon(a)k$ for all $a \in A$.

In general, cointegrals will not exist (see example below). However, if a left cointegral exists, then also a right cointegral exists. Indeed, by applying the antipode $S$ to a left cointegral $h$ we get a right cointegral.

The first main result in this section will be that if cointegrals exists, then they are unique (up to a scalar). We will also prove the existence of integrals in this case.

First, we look at some examples to illustrate this definition.

2.4. Example. (i) Let $G$ be a group and let $A$ denote the algebra of complex functions on $G$ with finite support. Then $A$ is a regular multiplier Hopf algebra if we define $\Delta$ by $(\Delta f)(p, q) = f(pq)$ whenever $f \in A$ and $p,q \in G$. The counit is given by $\epsilon(f) = f(e)$ where $e$ is the identity in $G$. If we let $h$ be the function which is 1 in $e$ and 0 everywhere else, then $h$ is both a left and a right cointegral.

(ii) If $A$ is a finite-dimensional Hopf algebra, then there always exist left and right cointegrals. They are left and right integrals on the dual Hopf algebra $A'$. It is possible to write down explicit formulas for these integral and cointegrals (see, e.g., [9, 13]). In general, unlike the previous example, left and right cointegrals may be different (see the example in [9] and compare also with the Example 3.10 in the next section).

(iii) Let $A$ be the algebra (with identity) generated by a single invertible element $a$. Then $A$ can be made into a Hopf algebra if we define $\Delta$ by $\Delta(a) = a \otimes a$. The counit is given by $\epsilon(a) = 1$. In this case cointegrals do not exist. Indeed, it is not possible to find a Laurent polynomial $p$ such that $ap(a) = p(a)$. Remark that in this example, we do have integrals. Let $\varphi(a^n) = 0$ if $n \neq 0$ and $\varphi(1) = 1$. Then $\varphi$ will be a left and a right integral. If we take the quotient of $A$ by imposing the extra condition $a^n = 1$ where $n \in \mathbb{N}$ and $n \geq 2$, we get a finite-dimensional Hopf algebra. A left and right cointegral is now given by $\sum_{k=1}^{n} a^k$.

If $(A, \Delta)$ is an algebraic quantum group of discrete type (see Definition 5.2 of [13]), then cointegrals exist by definition. At the end of this section, we will prove the converse: if cointegrals exist, then also integrals exist and so $(A, \Delta)$ is an algebraic quantum group of discrete type.

Let us now prove the most important step towards our main result in this section. It is the *faithfulness* of cointegrals. Remark that in [13], we also began with proving the faithfulness of integrals. In fact, the techniques that we use here are very similar to those used in [13].

Before we formulate our result, recall that for any $a \in A$ and $\omega \in A'$, we can define elements $(\iota \otimes \omega)\Delta(a)$ and $(\omega \otimes \iota)\Delta(a)$ in $M(A)$ (as we have seen in the Introduction).

2.5. PROPOSITION. *Let $h$ be a left cointegral in $A$. If $\omega$ is a linear functional on $A$ such that $(\iota \otimes \omega)\Delta(h) = 0$, then $\omega = 0$. Similarly, if $(\omega \otimes \iota)\Delta(h) = 0$, then $\omega = 0$.*

*Proof.* We will first show that

$$(1 \otimes a)\Delta(h) = (S(a) \otimes 1)\Delta(h)$$

for all $a \in A$. To prove this, take any $a, b, c \in A$. Then

$$\begin{aligned}
(b \otimes c)(1 \otimes a)\Delta(h) &= \sum_{(a)} \big(\epsilon(a_{(1)})b \otimes ca_{(2)}\big)\Delta(h) \\
&= \sum_{(a)} \big(bS(a_{(1)})a_{(2)} \otimes ca_{(3)}\big)\Delta(h) \\
&= \sum_{(a)} \big(bS(a_{(1)}) \otimes c\big)\Delta\big(a_{(2)}h\big) \\
&= \sum \big(\epsilon(a_{(2)})bS(a_{(1)}) \otimes c\big)\Delta(h) \\
&= (b \otimes c)\big(S(a) \otimes 1\big)\Delta(h)
\end{aligned}$$

and if we cancel $b$ and $c$, we get the desired formula.

Now assume that $\omega$ is a linear functional in $A$ such that $(\iota \otimes \omega)\Delta(h) = 0$. Then, using the previous formula, we obtain

$$\sum_{(h)} \omega\big(ah_{(2)}\big)h_{(1)} = 0$$

for all $a \in A$. Then also

$$\sum_{(h)} \omega\big(ah_{(3)}\big)bh_{(1)} \otimes cS\big(h_{(2)}\big) = 0$$

for all $a, b, c \in A$. Fix $b$ and $c$ and write

$$\sum bh_{(1)} \otimes cS\big(h_{(2)}\big) \otimes h_{(3)} = \sum p_i \otimes q_i \otimes r_i$$

with the $(q_i)$ linearly independent. We obtain that $\omega(ar_i)p_i = 0$ for all $i$ and all $a$. Replace $a$ by $q_i$ and take the sum over $i$ to get

$$\sum_{(h)} \omega\Big(cS\big(h_{(2)}\big)h_{(3)}\Big)bh_{(1)} = 0$$

and hence $\omega(c)bh = 0$. This holds for all $b$ and all $c$ and because $h \neq 0$ we get $\omega = 0$. The second statement is proven in a similar way from the property $\sum\omega(ah_{(1)})h_{(2)} = 0$ and using the inverse of the antipode. ∎

Of course, also right cointegrals are faithful. This can be shown in a similar fashion or by applying the antipode.

The formula $(1 \otimes a)\Delta(h) = (S(a) \otimes 1)\Delta(h)$ will be used on other occasions. For a right cointegral $k$ we have $\Delta(k)(a \otimes 1) = \Delta(k)(1 \otimes S(a))$. This can be shown in a similar way or by using the antipode.

Now we prove the following immediate consequence of the faithfulness of a cointegral.

2.6. PROPOSITION. *Suppose that h is a left cointegral. Then the maps*

$$\omega \to (\iota \otimes \omega)\Delta(h) \qquad \textit{and} \qquad \omega \to (\omega \otimes \iota)\Delta(h)$$

*are bijective from $H$ to $A$.*

*Proof.* We know already that these maps are injective (Proposition 2.5). Remark that, by the definition of $H$, the ranges are in $A$. Suppose, e.g., that the first map is not surjective. Then there is a $\rho \in A'$ so that $\rho((\iota \otimes \omega)\Delta(h)) = 0$ for all $\omega \in H$ but $\rho \neq 0$. By the definition of $H$ and the remark following Lemma 2.2, we can rewrite this as $\omega((\rho \otimes \iota)\Delta(h)) = 0$ for all $\omega \in H$. Therefore $(\rho \otimes \iota)\Delta(h) = 0$ and by Proposition 2.5 we have $\rho = 0$. This is a contradiction.

Similarly the second map is surjective. ∎

In the next section, we will modify the above map from $H$ to $A$ so that it becomes an $A$-module isomorphism (see Proposition 3.2). This result will yield the characterizations of the underlying algebra that we will obtain in Section 3.

Now, we prove another nice consequence of this result.

2.7. PROPOSITION. *If $h$ and $h'$ are two left cointegrals, there is a scalar $\lambda \in \mathbb{C}$ so that $h' = \lambda h$.*

*Proof.* Because $h$ is a left cointegral and $S(h')$ is a right cointegral we get

$$\begin{aligned}(1 \otimes h')\Delta(h) &= (S(h') \otimes 1)\Delta(h)\\ &= S(h') \otimes (\epsilon \otimes 1)\Delta(h)\\ &= S(h') \otimes h.\end{aligned}$$

Choose $e$ such that $h'e = h'$. Take $\omega \in H$ such that $(\omega \otimes \iota)\Delta(h) = e$. This is possible by the previous proposition. Then, applying $\omega \otimes \iota$ to the above formula we get $h' = \omega(S(h'))h$ and this proves the result. ∎

Of course, also right cointegrals will be unique (up to a scalar).

Remark that the uniqueness proof here is simpler than the one of integrals in [13]. It could be that also there the proof can be simplified using similar ideas.

By the uniqueness of the left cointegral, we obtain the existence of a homomorphism $\delta : A \to \mathbb{C}$ given by $ha = \delta(a)h$. As we will show later in this section, a regular multiplier Hopf algebra with cointegrals is an algebraic quantum group (in the sense of [13]). The homomorphism $\delta$ is nothing else but the modular element $\delta$ in $M(\hat{A})$, as introduced in [13].

The next step is to extend Proposition 2.6 to all of $A'$.

2.8. PROPOSITION. *Let $h$ be a left cointegral. Then the maps*

$$\omega \to (\iota \otimes \omega)\Delta(h) \qquad \textit{and} \qquad \omega \to (\omega \otimes \iota)\Delta(h)$$

*are bijective from $A'$ to $M(A)$.*

*Proof.* Again, also these maps are injective. To prove that, e.g., the first map is surjective, take any $x \in M(A)$. Define a linear map $\omega \in A'$ by

$$\omega((\rho \otimes \iota)\Delta(h)) = \rho(x)$$

whenever $\rho \in H$. This is possible because on the one hand, every element in $A$ can be uniquely written as $(\rho \otimes \iota)\Delta(h)$ and on the other hand, $\rho(x)$ is well-defined (and uniquely defined as we saw before). Then it is also clear that $\omega$ is linear.

From the definition we get $\rho(x) = \rho((\iota \otimes \omega)\Delta(h))$ and because $H$ separates points in $A$, we get $x = (\iota \otimes \omega)\Delta(h)$.

Similarly, we get that the second map is surjective. ∎

Now we are ready to obtain the main result in this section (Theorem 2.10 below). But before, we want to formulate and prove a result here, closely related with the previous proposition.

2.9. PROPOSITION. *Let $A$ be a regular multiplier Hopf algebra with cointegrals. Then $M(A)$ coincides with the dual space $H'$ of $H$.*

*Proof.* Define a map $\gamma : M(A) \to H'$ by $\gamma(x)(\omega) = \omega(x)$ where $x \in M(A)$ and $\omega \in H$ and where $\omega(x)$ is (well-)defined as before (see remark following Lemma 2.2).

It is fairly easy to see that $\gamma$ is injective. Indeed, if $\gamma(x) = 0$ then $\omega(x) = 0$ for all $\omega \in H$. In particular $\psi(xa) = 0$ for all $a \in A$ and $\psi \in A'$ and so $xa = 0$ for all $a$ and we must have $x = 0$.

To prove that $\gamma$ is surjective, take any $\Gamma \in H'$. Choose a left cointegral $h$. Define $\Psi \in A'$ by $\Psi((\iota \otimes \omega)\Delta(h)) = \Gamma(\omega)$ where $\omega \in H$. This is well-defined by Proposition 2.6. Then, let $x = (\Psi \otimes \iota)\Delta(h)$. We get $x \in M(A)$ and it is easily seen that

$$\begin{aligned}\gamma(x)(\omega) &= \omega((\Psi \otimes \iota)\Delta(h)) \\ &= \Psi((\iota \otimes \omega)\Delta(h)) \\ &= \Gamma(\omega)\end{aligned}$$

so that $\gamma(x) = \Gamma$. ∎

2.10. THEOREM. *If $(A, \Delta)$ is a regular multiplier Hopf algebra with cointegrals, then $(A, \Delta)$ also has integrals.*

*Proof.* Let $h$ be a left cointegral. Choose $\varphi \in A'$ so that $(\iota \otimes \varphi)\Delta(h) = 1$. This is possible by Proposition 2.8. We will now show that $\varphi$ is a left integral. Take $a \in A$ and write $a = (\omega \otimes \iota)\Delta(h)$ with $\omega \in H$. This is possible by Proposition 2.6. Now write $\omega = \psi(\cdot\, b)$ so that $\varphi(a) = \psi(b)$. For any $c \in A$ we get

$$\begin{aligned}\sum_{(a)} \varphi(a_{(2)})ca_{(1)} &= \sum_{(h)} \varphi(h_{(3)})\psi(h_{(1)}b)ch_{(2)} \\ &= \psi(b)c = \varphi(a)c\end{aligned}$$

and this means that $\varphi$ is left invariant. ∎

Similarly, we will get a right integral $\psi$ if $(\psi \otimes \iota)\Delta(h) = 1$.

The proof of Theorem 2.10 is very similar to the proof of the existence of integrals on discrete quantum groups as we can find in [11]. In fact, the one that we have here is slightly simpler and certainly more general. It is also simpler than the original proof given by Effros and Ruan in [4].

2.11. *Remark*. It follows from this theorem that a regular multiplier Hopf algebra with cointegrals is an algebraic quantum group of discrete type (in the sense of Definition 5.2 of [13]). Of course, conversely, every algebraic quantum group of discrete type has cointegrals (by definition). So, the algebraic quantum groups of discrete type are precisely the regular multiplier Hopf algebras with cointegrals. Therefore, we also will use the term regular multiplier Hopf algebra of discrete type.

In [13], we have shown that integrals are unique if they exist. Here, this result follows very easily from Proposition 2.8. Indeed, if $\varphi$ and $\varphi'$ are left integrals, then for any left cointegral $h$ we get $(\iota \otimes \varphi)\Delta(h) = \varphi(h)1$ and $(\iota \otimes \varphi')\Delta(h) = \varphi'(h)1$ and hence, if we assume $\varphi(h) = \varphi'(h)$ we get $\varphi = \varphi'$.

We will finish this section with some consequences about the space $H$.

2.12. Proposition. *If $(A, \Delta)$ is of discrete type, then $H = \hat{A}$ where $\hat{A}$ is defined by*

$$\{\varphi(\cdot\, a) \mid a \in A\}$$

*and $\varphi$ is any left integral (see [13, 4.1]). Therefore $H$ is a Hopf algebra (with product and coproduct dual to the coproduct and product in $A$).*

*Proof.* Take any left integral $\varphi$ and any left cointegral $h$. For any $a$ we have

$$\begin{aligned}(\iota \otimes \varphi)((1 \otimes a)\Delta(h)) &= (\iota \otimes \varphi)(S(a) \otimes 1)\Delta(h) \\ &= \varphi(h)S(a).\end{aligned}$$

Now, by Proposition 2.6, there is a $\omega \in H$ such that $\varphi(h)S(a) = (\iota \otimes \omega)\Delta(h)$ and again by the same proposition (or by the faithfulness of $h$ if you like), we must have that $\omega = \varphi(\cdot\, a)$. So $H \subseteq \hat{A}$. The other inclusion is obvious.

Now, it is shown in Section 4 of [13] that $\hat{A}$ is a multiplier Hopf algebra. Observe that $\epsilon \in \hat{A}$ and that it is the identity in $\hat{A}$. So, $\hat{A}$ is actually a Hopf algebra. ∎

Remark that in this case, $H$ is a Hopf algebra with integrals (given by the cointegrals in $A$). Also $H$ is a multiplier Hopf algebra of compact type (or an algebraic quantum group of compact type) in the sense of Definition 5.2 of [13].

Remark further that the invariance of $H$ under $S$ implies that also $H = \{\omega(a\,\cdot\,) \mid a \in A,\ \omega \in A'\}$, a result which is not so obvious. This equality of subsets of $A'$ is similar to the weak K.M.S. property for integrals (see [13]). We will use this equality (true for multiplier Hopf algebras of discrete type) in the next section.

## 3. THE UNDERLYING ALGEBRA OF A MULTIPLIER HOPF ALGEBRA OF DISCRETE TYPE

Let $(A, \Delta)$ be a regular multiplier Hopf algebra. In this section we will give some necessary and sufficient conditions on the algebra $A$ for $(A, \Delta)$ to be of discrete type.

In Section 2, in the beginning, we proved that given a finite set of elements $a_1, a_2, \ldots, a_n$ in a regular multiplier Hopf algebra, there exist elements $e$ and $f$ in $A$ satisfying $ea_k = a_k f = a_k$ for all $k$. We mentioned that we can take $e = f$ when $A$ is an algebraic quantum group. We will now obtain an even stronger result for algebras of discrete type.

3.1. PROPOSITION. *Assume that $(A, \Delta)$ is of discrete type. For any finite set $a_1, a_2, \ldots, a_n$ of elements in $A$, there is an idempotent $e$ in $A$ such that $ea_k = a_k e = a_k$ for all $k$.*

*Proof.* Consider the dual Hopf algebra $H$ (see Proposition 2.12). Then $H$ is a co-Frobenius Hopf algebra by [6]. Also $A$ is the rational dual $H^{\square}$ of $H$ in the sense of Sweedler. In [2] it is shown that there exists a set $\{e_i\}_{i \in I}$ of mutually orthogonal idempotents in $A$ such that

$$A = H^{\square} = \sum_{i \in I} e_i A = \sum_{i \in I} A e_i = {}^{\square}H.$$

It follows that $A = \sum_{i,j} e_i A e_j$. Now, let $a_1, a_2, \ldots, a_n$ be a finite subset of $A$. Then each of these elements is a finite sum of elements of the form $e_i a e_j$ for some $a \in A$ and $i, j \in I$. Clearly, if we set $e = \sum_{i \in I_0} e_i$ where we take in $I_0$ all the indices that occur in these expressions for all $a_k$, we get an idempotent $e$ in $A$ such that $ea_k = a_k e = a_k$ for all $k$. ∎

Remark that Proposition 3.1 says that $A$ has *local units*. Also remark that this result is obviously true for the more restricted case of a discrete quantum group. Then $A$ is assumed to be a direct sum of full matrix algebras [4, 11].

Recall that we consider the dual space $A'$ as a left $A$-module by $a\omega = \omega(\cdot\, a)$ whenever $a \in A$ and $\omega \in A'$ and that we denote the submodule $AA'$ by $H$. In general, we have by Lemma 2.2 that $H = \{\omega(\cdot\, a) \mid a \in A, \omega \in A'\}$. Moreover, when $(A, \Delta)$ is of discrete type, then also $H = \{\omega(a\,\cdot\,) \mid a \in A, \omega \in A'\}$.

Now, we have the following result.

3.2. PROPOSITION. *Assume that* $(A, \Delta)$ *is of discrete type. Let h be a left cointegral in A. Define a map* $\Gamma : H \to A$ *by* $\Gamma(\omega) = (\iota \otimes S(\omega))\Delta(h)$. *Then* $\Gamma$ *is a A-module isomorphism.*

*Proof.* First, remark that the antipode $S$ is a bijection of $H$ (see the remark following Proposition 2.12). Therefore, it follows from Proposition 2.6 that $\Gamma$ is a vector space isomorphism. To show that it is a $A$-module isomorphism, take any $a \in A$ and use $(a \otimes 1)\Delta(h) = (1 \otimes S^{-1}(a))\Delta(h)$ to obtain

$$\begin{aligned} a\Gamma(\omega) &= (\iota \otimes S(\omega))\big((1 \otimes S^{-1}(a))\Delta(h)\big) \\ &= (\iota \otimes S(a\omega))\Delta(h) = \Gamma(a\omega) \end{aligned}$$

for all $\omega \in H$. ∎

Remark that, using Proposition 2.8, we also have that the map $\omega \to (\iota \otimes S(\omega))\Delta(h)$ is a $A$-module isomorphism from $A'$ to $M(A)$.

Recall that a finite-dimensional algebra $A$ (with identity) is called a Frobenius algebra if there is an $A$-module isomorphism from $A$ to $A'$ (both considered as left $A$-modules), see, e.g., [6]. In fact, this is equivalent to the existence of a faithful linear functional (as defined in the Introduction). So, for finite-dimensional Hopf algebras, it follows automatically that the underlying algebras are Frobenius. This result extends the same result for group algebras. See also [5]. The situation is different in the infinite dimensional case. Here we get an $A$-module isomorphism from $A$ to $AA'$ for an algebra which is not only possibly infinite-dimensional but also may fail to have an identity. It makes sense, however, to call an algebra a *Frobenius algebra* if it has the above property.

In the next proposition, we also prove the converse and we obtain our first characterization.

3.3. THEOREM. *Let* $(A, \Delta)$ *be a regular multiplier Hopf algebra. Then* $(A, \Delta)$ *is of discrete type if and only if A is a Frobenius algebra.*

*Proof.* We have shown already one direction. Now assume that $A$ is Frobenius and that $\Gamma : H \to A$ is a $A$-module isomorphism. Let $h = \Gamma(\epsilon)$. Then $ah = \Gamma(a\epsilon) = \epsilon(a)\Gamma(a) = \epsilon(a)h$. Therefore $h$ is a left cointegral and the result follows from Theorem 2.10. ∎

For the next characterization, we will still work with the $A$-module isomorphism $\Gamma$. The results will involve the notion of annihilator of an ideal. Let us recall this in the next definition.

3.4. DEFINITION. Let $I$ be a left ideal of $A$. We call the set of elements $b \in A$ such that $ab = 0$ for all $a \in I$ the *right annihilator* of $I$ and we denote it by $r(I)$. Similarly, we define the *left annihilator* of a right ideal $J$ as the set of elements $b \in A$ such that $ba = 0$ for all $a \in J$ and we denote it by $l(J)$.

Remark that $r(I)$ is a right ideal and that $l(J)$ is a left ideal.

Now we can prove the following lemma.

3.5. LEMMA. *Let $A$ be Frobenius. Let $\Gamma$ be a $A$-module isomorphism from $H$ to $A$. For any left ideal $I$ of $A$ we have that $\Gamma(I^{\perp}) = r(I)$ where $I^{\perp} = \{\omega \in H \mid \omega(a) = 0$ for all $a \in I\}$.*

*Proof.* Let $\omega \in H$ and let $a \in I$. Then $a\Gamma(\omega) = \Gamma(a\omega)$ so that $\Gamma(\omega) \in r(I)$ if and only if $\omega(xa) = 0$ for all $x \in A$ and $a \in I$. But as $I$ is a left ideal, this precisely means that $\omega$ is 0 on $I$. ∎

We can now prove the second characterization.

3.6. THEOREM. *Let $(A, \Delta)$ be a regular multiplier Hopf algebra. Then the following are equivalent*:

(i) *$(A, \Delta)$ is of discrete type,*

(ii) *any proper left ideal has a non-zero right annihilator,*

(iii) *every proper two-sided ideal has a non-zero right annihilator.*

*Proof.* (i) ⇒ (ii). If $(A, \Delta)$ is of discrete type, we know that $A$ is Frobenius and, by the lemma, we must show that $I^{\perp}$ is non-trivial when $I$ is a proper left ideal. So, let $I$ be a left ideal of $A$ and assume that $I \neq A$. Choose $\omega \in A'$ so that $\omega \neq 0$ but $\omega \mid I = 0$. Consider any $b \in A$ and let $\rho = \omega(b\,\cdot\,)$. Still $\rho \mid I = 0$. If, however, $\rho = 0$ for all $b$, then $\omega(ba) = 0$ for all $a, b \in A$ and because $A^2 = A$, we would get $\omega = 0$. So for some $b$, we have $\rho \neq 0$ and $\rho \in I^{\perp}$.

(ii) ⇒ (iii). This is obvious.

(iii) ⇒ (i). Consider the kernel $I$ of the counit $\epsilon$. This is a proper two-sided ideal. Take a non-zero element $k$ in the right annihilator of $I$. Because $ab - \epsilon(a)b \in I$, we get $kab = \epsilon(a)kb$ for all $a, b \in A$. If we cancel $b$, we get $ka = \epsilon(a)k$ for all $a$ and $k$ is a right cointegral. Now, the result follows from Theorem 2.10. ∎

Of course, in the previous result, we can interchange left and right.

Recall that an algebra $A$ satisfying (iii) is called a *Kasch algebra* (see [7]).

Finally, we will obtain our last characterization. Recall the following definition (see also [7]).

3.7. DEFINITION. An algebra $A$ is called *quasi-Frobenius* if for any left ideal $I$ and any right ideal $J$ we have

$$lr(I) = I \qquad \text{and} \qquad rl(J) = J.$$

Then we have the following.

3.8. THEOREM. *Let $(A, \Delta)$ be a regular multiplier Hopf algebra. Then $(A, \Delta)$ is of discrete type if and only if $A$ is quasi-Frobenius.*

*Proof.* Suppose first that $(A, \Delta)$ is of discrete type. Choose a left cointegral $h$ and let $\Gamma$ be the $A$-module isomorphism from $H$ to $A$ defined as in Proposition 3.2 by $\Gamma(\omega) = (\iota \otimes S(\omega))\Delta(h)$. Let $I$ be a left ideal. It is clear that $I \subseteq lr(I)$. If these two left ideals are different, we can choose $\omega \in A'$ such that $\omega$ is zero on $I$ but not zero on $lr(I)$. And again, as in the proof of Theorem 3.6, we can assume $\omega \in H$. Therefore, it remains to show that $\omega$ is zero on $lr(I)$ when $\omega \in I^{\perp}$. So assume $\omega \in I^{\perp}$ and take $a \in lr(I)$. Because of Lemma 3.5 we get $a\Gamma(\omega) = 0$. Hence $a\omega = 0$. But this implies $\omega(a) = 0$ (see Section 2). This completes the proof of the fact that $rl(I) = I$ for any left ideal. By applying the antipode, we also get $lr(J) = J$ for any right ideal.

Now conversely, suppose that $lr(I) = I$ for any left ideal $I$. If we take for $I$ a proper left ideal, it will follow that $r(I)$ is non-zero. Hence, by Theorem 3.6 we have that $(A, \Delta)$ is of discrete type. ∎

We will now illustrate these results by some examples. But before we do this, remark that all these properties are valid for finite-dimensional Hopf algebras (as they are of discrete type).

3.9. EXAMPLES. (i) First take the example of the algebra $A$ of complex functions with finite support on a group $G$ with comultiplication $\Delta$ given by $(\Delta f)(p, q) = f(pq)$ when $f \in A$ and $p, q \in G$. In this case, $H$ is given by linear mappings $\omega$ of the form $\omega(f) = \sum f(p)g(p)$ where also $g$ is a complex function on $G$ with finite support. In fact, $H$ is the group Hopf algebra $\mathbb{C}G$. The function $h$ defined as 1 in the identity $e$ of $G$ and 0 everywhere else is a cointegral. The map $\Gamma$, with the identification of $H$ with $A$ as above, is given by

$$\Gamma(g)(p) = \sum_q h(pq)g(q^{-1}) = g(p).$$

So $\Gamma$ becomes the identity map and we obviously have a Frobenius algebra. Ideals in $A$ are given by subsets $G_0$ of $G$. If $G_0$ is a subset of $G$ and $I$ is the ideal of functions with finite support in $G_0$, then clearly $r(I)(= l(I))$ is the ideal of functions with finite support in $G \setminus G_0$. Then also the two other characterizations are easily verified.

(ii) Not so different from the above example is a discrete quantum group (in the sense of [4, 11]). Here the underlying algebra is a direct sum of matrix algebras and it is easy to see that also in this case, we have the Frobenius and the quasi-Frobenius property.

So far, we have two more or less trivial examples. Next, we will consider another (less trivial) example, similar to the one we have obtained in Proposition 5.6 of [13].

3.10. EXAMPLE. (i) First, consider the algebra $A$ spanned by elements $\{e_p b^q \mid p \in \mathbb{Z} \text{ and } q = 0, 1, 2, \ldots\}$ and where the elements $e_p$ and $b$ satisfy the relations $e_p e_q = \delta(p, q) e_p$ and $be_p = e_{p+1} b$. Choose any $\lambda \in \mathbb{C} \setminus \{0\}$. Define $a$ in $M(A)$ by $a = \sum_{k \in \mathbb{Z}} \lambda^k e_k$. Then define a comultiplication $\Delta$ on $A$ by

$$\Delta(e_p) = \sum_{k \in \mathbb{Z}} e_k \otimes e_{p-k}$$

$$\Delta(b) = a \otimes b + b \otimes a^{-1}.$$

Remark that these infinite sums are well-defined in the "strict topology" on the multiplier algebra (i.e., when one multiplies with elements of the algebra, one gets finite sums). One can check that $(A, \Delta)$ is a regular multiplier Hopf algebra. The counit is given by $\epsilon(e_p) = \delta(p, 0)$ (and hence $\epsilon(a) = 1$) and $\epsilon(b) = 0$. The antipode is given by $S(e_p) = e_{-p}$ (and hence $S(a) = a^{-1}$) and $S(b) = -\lambda^{-1} b$. One can verify that no integrals and no cointegrals exist for this example.

(ii) Now we will consider a quotient of the above algebra. Assume that $\lambda$ is a root of unity (different from 1 and $-1$) and that $n$ is the smallest natural number such that $\lambda^{2n} = 1$ (so $n \geq 2$). Take the quotient of the algebra $A$ above by imposing the extra condition $b^n = 0$. Denote this quotient by $B$. By the choice of $\lambda$ and $n$, one can verify that also $(a \otimes b + b \otimes a^{-1})^n = 0$ in $M(B \otimes B)$. Therefore, $\Delta$ is also well-defined on the quotient algebra $B$. Still, $B$ will be a regular multiplier Hopf algebra. The counit and the antipode are given by the same formulas. Now, it is easy to check that $e_0 b^{n-1}$ is a left cointegral and that $e_{n-1} b^{n-1}$ is a right cointegral (observe that they are different). A left integral $\varphi$ is given by $\varphi(e_p b^{n-1}) = \lambda^{p(n-1)}$ and $\varphi(e_p b^q) = 0$ if $q \neq n-1$.

It is tedious to calculate the module isomorphism $\Gamma$ between $A$ and $H$. To illustrate the quasi-Frobenius property, we will only consider a very special left ideal since also here, calculations become rather involved. Take, e.g., the left ideal $I$ spanned by the elements $e_p b^{n-1}$ with $p \geq 0$. The right annihilator $r(I)$ can be seen to be spanned by the elements $e_p$ with $p < -n + 1$ and the elements $e_p b^q$ with $q > 0$ and any $p$. One can verify that the left annihilator of this right ideal is again $I$.

## REFERENCES

1. E. Abe, ‘‘Hopf Algebras,’’ Cambridge Univ. Press, Cambridge, UK, 1977.
2. H. Allen and D. Trushin, Coproper coalgebras, *J. Algebra* **54** (1978), 203–215.
3. B. Drabant and A. Van Daele, Pairing and the quantum double of multiplier Hopf algebras, preprint, K. U. Leuven, 1996.
4. E. Effros and Z.-J. Ruan, Discrete quantum groups. I. The Haar measure, *Internat. J. Math.* **5** (1994), 681–723.
5. H. Larson and E. Sweedler, An associative orthogonal bilinear form for Hopf algebras, *Amer. J. Math.* **91** (1969), 75–94.
6. I.-P. Lin, Semiperfect algebras, *J. Algebra* **49** (1977), 357–373.
7. B. Stenstrom, ‘‘Rings of Quotients,’’ Springer-Verlag, Berlin, 1975.
8. E. M. Sweedler, ‘‘Hopf Algebras,’’ Benjamin, Elmsford, NY, 1969.
9. A. Van Daele, The Haar measure on finite quantum groups, *Proc. Amer. Math. Soc.* **125** (1997), 3489–3500.
10. A. Van Daele, Multiplier Hopf algebra, *Trans. Amer. Math. Soc.* **342** (1994), 917–932.
11. A. Van Daele, Discrete quantum groups, *J. Algebra* **180** (1996), 431–444.
12. A. Van Daele, Multiplier Hopf algebras and duality, *in* ‘‘Proceedings of the Workshop on Quantum Groups and Quantum Spaces, Warsaw, November 1995,’’ Banach Center Publications, Vol. 40, Polish Acad. Sci., Warsaw, 1997.
13. A. Van Daele, An algebraic framework for group duality, *Adv. in Math.* **140** (1998), 323–366.
14. B. Drabant, A. Van Daele, and Y. Zhang, Actions of multiplier Hopf algebras, *Comm. Algebra*, in press.